

Weak Coupling Chambers in $\mathcal{N} = 2$ BPS Quiver Theory

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Abstract

Using recent results on BPS quiver theory, we develop a group theoretical method to describe the quiver mutations encoding the quantum mechanical duality relating the spectra of distinct quivers. We illustrate the method by computing the BPS spectrum of the infinite weak chamber of some examples of $\mathcal{N} = 2$ supersymmetric gauge models without and with quark hypermultiplets.

Key words: Electric magnetic duality in $N = 2$ QFT₄, BPS quiver theory, Quiver mutations.

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1 Introduction

Recently a BPS quiver theory has been proposed in [1, 2] to build the full set of BPS spectra in 4D $\mathcal{N} = 2$ supersymmetric quantum field theory (QFT₄). This quiver model is based on quantum mechanical dualities encoded by *quiver mutations* and relating distinct quivers of the theory.

In this paper, we develop the group structure property¹ of the set quiver mutations in $\mathcal{N} = 2$ supersymmetric quantum theory with $SU(2)$ gauge symmetry; and use this remarkable property to approach the complexity of the infinite BPS weak coupling chambers of these supersymmetric theories. To illustrate the key idea of our method, we consider below the weak coupling chambers of 3 particular supersymmetric QFTs: (i) the weak coupling chamber of the $\mathcal{N} = 2$ supersymmetric pure $SU(2)$ gauge theory spontaneously broken down to $U(1)$. The BPS spectrum of this model has been studied in [1, 2]; see also [3] and refs therein; but here we will re-derive this infinite spectrum from a group theory approach. (ii) the weak coupling chambers of the $\mathcal{N} = 2$ supersymmetric $SU(2)$ gauge theory with one quark hypermultiplet; and (iii) with two quark hypermultiplets. We end this paper by a conclusion and a comment on the method including the link between the quiver mutation symmetries and the strong/weak coupling chambers of $\mathcal{N} = 2$ QFT₄s.

2 Weak coupling chamber of $SU(2)$ model

We begin by recalling that the low energy $\mathcal{N} = 2$ supersymmetric $SU(2)$ QFT has a monopole \mathfrak{M} and a dyon \mathfrak{D} believed to be two elementary BPS states of the $\mathcal{N} = 2$ superalgebra with central charges [4, 5, 6, 7]; see also [8, 9, 10] for a review. Let denote by X and Y the complex central charges of these two BPS states; and by γ_1 and γ_2 their electric magnetic (EM) vectors. These central charges and EM charge vectors are related;

¹the set quiver mutations for higher dimensional gauge symmetries has a groupoid structure [11].

and play an important role in the study of the BPS spectra of this supersymmetric gauge theory; the masses of \mathfrak{M} and \mathfrak{D} are proportional the absolute values $|X|$ and $|Y|$; and the arguments $\arg X$, $\arg Y$ for distinguishing the two possible BPS chambers of the $SU(2)$ gauge theory namely: (a) $\arg X < \arg Y$ describing the strong coupling chamber of the supersymmetric $SU(2)$ QFT; and (b) $\arg X > \arg Y$ for the weak coupling one. The first one is finite as it contains the two elementary BPS states and their CPT conjugates. The second BPS chamber is very rich; it is infinite and remarkably built in the framework of BPS quiver theory that we consider here below.

2.1 BPS spectrum

This spectrum has been explicitly built in [1, 2] by using the BPS quiver theory. There, the BPS states of the weak coupling limit of the $\mathcal{N} = 2$ supersymmetric $SU(2)$ QFT is obtained from the primitive quiver \mathfrak{Q}_0 by performing two kinds of quiver mutations: left mutations and right ones. Under left mutations of \mathfrak{Q}_0 , made of the monopole \mathfrak{M} and the dyon \mathfrak{D} , we get an infinite set of BPS and anti-BPS states with respective electric-magnetic (EM) charges $\gamma_{left}^{(n)}$, $\gamma'_{left}{}^{(n)}$ as follows

$$\begin{aligned}\gamma_{left}^{(n)} &= (n+1)\gamma_1 + n\gamma_2 \\ \gamma'_{left}{}^{(n)} &= -n\gamma_1 - (n+1)\gamma_2\end{aligned}\tag{2.1}$$

with n an arbitrary positive integer; $n \geq 0$. Under the right mutation, we also obtain an infinite BPS and anti-BPS states with EM charges $\gamma_{right}^{(n)}$, $\gamma'_{right}{}^{(n)}$ given by,

$$\begin{aligned}\gamma_{right}^{(n)} &= n\gamma_1 + (n+1)\gamma_2 \\ \gamma'_{right}{}^{(n)} &= -(n+1)\gamma_1 - n\gamma_2\end{aligned}\tag{2.2}$$

The left and right spectra are related by interchanging the role of the γ_1 and γ_2 ; or by performing $\gamma_i \rightarrow -\gamma_i$; the second link may be also interpreted in terms of CPT invariance of the BPS chamber.

Below we show that this spectrum can be also derived by using a group theory method; this approach has the property of giving a group theoretical interpretation of quiver mutations; in particular the right mutations are precisely the inverse of the left ones. It also has the power to deal with the complexity of weak chamber of supersymmetric gauge theories without and with matter including higher dimensional gauge symmetries.

2.2 Group theory approach

The key idea of our group theoretical approach that we develop below relies on the two following basic things: **(1)** think about the primitive quiver \mathfrak{Q}_0 , of the BPS quiver theory of $\mathcal{N} = 2$ supersymmetric $SU(2)$ gauge model, as given by the pair $(\Upsilon^0, \mathcal{A}^0)$ consisting of a vector Υ^0 and an intersection matrix \mathcal{J}^0 as follows

$$\Upsilon^0 = \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \quad , \quad \mathcal{J}^0 = \Upsilon^0 \circ \Upsilon^0 \quad (2.3)$$

with $\Upsilon \circ \Upsilon$ standing for the electric-magnetic product. **(2)** realize the quiver mutations of the primitive \mathfrak{Q}_0 in terms of a set $\{\mathcal{M}_m\}$ of discrete rotations in the space of the EM charges as follows

$$\Upsilon^{(m)} = \mathcal{M}_m \Upsilon^0 \quad , \quad \mathcal{J}^{(m)} = \mathcal{M}_m \mathcal{J}^0 \mathcal{M}_m^T \quad (2.4)$$

with m an arbitrary positive integer. In this view, a generic mutation \mathcal{M}_m maps the primitive \mathfrak{Q}_0 into a mutated one \mathfrak{Q}_m given by a pair $(\Upsilon^{(m)}, \mathcal{J}^{(m)})$. In the above relations, the mutation transformations are given by invertible 2×2 matrices with integer entries; and to compare with the result of [1, 2], these \mathcal{M}_m 's have to encode both the left \mathcal{M}_m^{left} and right \mathcal{M}_m^{right} mutations of (2.1-2.2). To that purpose, let us refer to the matrices \mathcal{M}_m^{left} and \mathcal{M}_m^{right} respectively as \mathcal{L}_m and \mathcal{R}_m ; and the set of mutations of the weak chamber of this BPS quiver theory as

$$\mathcal{G}_{weak}^{su_2} = \{\mathcal{L}_m, \mathcal{R}_m; m \in \mathbb{N}\} \subseteq GL(2, \mathbb{Z}) \quad (2.5)$$

From the explicit computation of the \mathcal{L}_m and right \mathcal{R}_m mutations, we learn that their expressions depend on the parity of the positive integer m . This is why it is helpful to split the \mathcal{L}_m 's and the \mathcal{R}_m 's using even and odd integers like

$$\mathcal{L}_m = (\mathcal{L}_{2k}, \mathcal{L}_{2k+1}) \quad , \quad \mathcal{R}_m = (\mathcal{R}_{2k}, \mathcal{R}_{2k+1}) \quad (2.6)$$

with k an arbitrary positive integer. Straightforward calculations using quiver mutation rules of the quiver BPS theory show that the explicit expressions of the mutations matrices \mathcal{L}_{2k} , \mathcal{L}_{2k+1} as well as \mathcal{R}_{2k} , \mathcal{R}_{2k+1} are as follows:

left sector	right sector
$\mathcal{L}_{2k} = \begin{pmatrix} 1+2k & 2k \\ -2k & 1-2k \end{pmatrix}$	$\mathcal{R}_{2k} = \begin{pmatrix} 1-2k & -2k \\ 2k & 1+2k \end{pmatrix}$
$\mathcal{L}_{2k+1} = \begin{pmatrix} -1-2k & -2k \\ 2k+2 & 1+2k \end{pmatrix}$	$\mathcal{R}_{2k+1} = \begin{pmatrix} 2k+1 & 2k+2 \\ -2k & -2k-1 \end{pmatrix}$

(2.7)

These mutation matrices obey a set of remarkable properties such as the involutions $(\mathcal{L}_{2k+1})^2 = I_{id}$ and $(\mathcal{R}_{2k+1})^2 = I_{id}$, for any integer k ; and moreover

$$\begin{aligned}\det \mathcal{L}_{2k} &= \det \mathcal{R}_{2k} = +1, \quad \forall k \\ \det \mathcal{L}_{2k+1} &= \det \mathcal{R}_{2k+1} = -1, \quad \forall k\end{aligned}\tag{2.8}$$

showing that $\mathcal{L}_{2k}, \mathcal{L}_{2k+1}, \mathcal{R}_{2k}, \mathcal{R}_{2k+1}$ are invertible matrices; and therefore the set $\mathcal{G}_{weak}^{su_2}$ form a subgroup of $GL(2, \mathbb{Z})$. To establish the relations (2.7), one uses the representation

$$\mathcal{L}_{2k} = (BA)^k, \quad \mathcal{R}_{2k} = (AB)^k\tag{2.9}$$

$$\mathcal{L}_{2k+1} = A\mathcal{L}_{2k}, \quad \mathcal{R}_{2k+1} = \mathcal{R}_{2k}A$$

showing that the infinite set $\mathcal{G}_{weak}^{su_2}$ of quiver mutations is indeed a subgroup of $GL(2, \mathbb{Z})$. This set is generated by two reflections A and B given by the triangular matrices

$$A = \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}\tag{2.10}$$

satisfying the usual reflection property $A^2 = B^2 = I_{id}$ and allow to build all possible elements of $\mathcal{G}_{weak}^{su_2}$ as in (2.9). Our way of doing gives then a group theoretical realization of the quantum mechanical duality of the BPS quiver theory. The $\mathcal{G}_{weak}^{su_2}$ group permits to map the primitive quiver \mathfrak{Q}_0 , thought of as in (2.3), to any dual quiver \mathfrak{Q}_m given by (2.4). It allows to get directly the BPS spectrum of the weak chamber since the EM charge vectors of the BPS states are precisely given by the row of the matrices (2.7). We have:

	left sector		right sector
$\gamma_1^{(2k)}$	$= \gamma_1 + 2kw$		$\gamma_1^{(2k)} = \gamma_1 - 2kw$
$\gamma_2^{(2k)}$	$= \gamma_2 - 2kw$		$\gamma_2^{(2k)} = \gamma_2 + 2kw$
$\gamma_1^{(2k+1)}$	$= \gamma_2 - (2k+1)w$		$\gamma_1^{(2k+1)} = \gamma_2 + (2k+1)w$
$\gamma_2^{(2k+1)}$	$= \gamma_1 + (2k+1)w$		$\gamma_2^{(2k+1)} = \gamma_1 - (2k+1)w$

(2.11)

with $w = \gamma_1 + \gamma_2$ giving the EM charge of the W-boson vector particle. From the above results, we learn that the BPS spectrum of the weak coupling chamber of the supersymmetric $SU(2)$ gauge theory is infinite; and moreover the number of BPS states grows linearly with the positive integer k . If defining the asymptotic limit of the mutation

matrices \mathcal{M}_n and by the regularized relation $\mathcal{M}_\infty = \lim_{n \rightarrow \infty} \left(\frac{1}{n}\mathcal{M}_n\right)$ we find

$$\mathcal{M}_\infty = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \quad , \quad A\mathcal{M}_\infty = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \quad (2.12)$$

with $\det \mathcal{M}_\infty = 0$ and similarly $\det A\mathcal{M}_\infty = 0$. These singular asymptotic limits describe the BPS particles with charges $\pm w$. These are vector particles having only electric charges; but no magnetic ones; they are associated with $\mathcal{N} = 1$ massive W^\pm vector multiplets.

3 Adding a hypermultiplet

In this section, we extend the above $\mathcal{N} = 2$ supersymmetric $SU(2)$ gauge model to include a quark hypermultiplet with a unit flavor charge under the $U_f(1)$ symmetry; and use the mutation group method we have developed above to compute the BPS spectrum of the weak coupling chamber of this theory. The elementary BPS particles are then given by the monopole \mathfrak{M} , the dyon \mathfrak{D} and the quark hypermultiplet \mathfrak{H} with respective electric-magnetic charges $\gamma_1, \gamma_2, \gamma_3$ as follows

$$\gamma_1 = \begin{pmatrix} 0 \\ 1 \\ \frac{1}{2} \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 1 \\ -1 \\ \frac{1}{2} \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad (3.1)$$

The third entry of these vectors is the charge under the $U_f(1) \simeq SO_f(2)$ flavor symmetry; see also [2] and [3] for details regarding the obtention of the primitive quiver $\mathfrak{Q}_0(N_f = 1)$ whose graph is given fig 1-(b); and for a comment on the flavor charges of the monopole and the dyon.

3.1 Weak coupling chamber

Extending the group theoretical approach described in previous section to the present case, we can compute directly the BPS spectrum of this gauge theory. This spectrum is given by the infinite set

$$\begin{aligned} &\gamma_1^{(n)}, \quad \gamma_2^{(n)}, \quad \gamma_3^{(n)} \\ &\gamma_1'^{(n)}, \quad \gamma_2'^{(n)}, \quad \gamma_3'^{(n)} \end{aligned} \quad (3.2)$$

with $\{\gamma_i^{(n)}\}, \{\gamma_i'^{(n)}\}, n \in \mathbb{N}$; describing respectively the left and the right sectors. These sectors are obtained separately by performing left mutations and then right ones. As in eq(2.11); it turns out that the BPS spectrum is given as well by several sequences. More

precisely, we have 6 families given by the classes of $\mathbb{Z}/6\mathbb{Z}$; and so we have to distinguish different sub-chambers according to the values of the integer n . We have:

(i) *the class $n=6k$*

This sequence describe a BPS sub-chamber of the weak coupling limit; the BPS and anti-BPS states in this sub-chamber are obtained by performing left and right quiver mutations. Under $6k$ successive elementary left mutations of the primitive quiver \mathfrak{Q}_0 , we obtain

$$\begin{aligned}\gamma_1^{(6k)} &= (1+3k)\gamma_1 + 3k(\gamma_2 + \gamma_3) = (6k, 1, \frac{1}{2}) \\ \gamma_2^{(6k)} &= (1-3k)\gamma_2 - 3k(\gamma_1 + \gamma_3) = (1-6k, -1, \frac{1}{2}) \\ \gamma_3^{(6k)} &= \gamma_3 = (1, 0, -1)\end{aligned}\tag{3.3}$$

with k a positive integer. Similarly under $6k$ successive elementary right mutations; we get

$$\begin{aligned}\gamma_1'^{(6k)} &= (1-3k)\gamma_1 - 3k(\gamma_2 + \gamma_3) = (-6k, 1, \frac{1}{2}) \\ \gamma_2'^{(6k)} &= (1+3k)\gamma_2 + 3k(\gamma_1 + \gamma_3) = (6k+1, -1, \frac{1}{2}) \\ \gamma_3'^{(6k)} &= \gamma_3 = (1, 0, -1)\end{aligned}\tag{3.4}$$

Notice that this sub-chamber contains the elementary BPS states for $k=0$; and the two sets (3.3-3.4) are related by the change $k \leftrightarrow -k$. Notice also that the sub-chamber is not CPT invariant.

(ii) *the class $n=6k+1$*

Under $6k+1$ successive elementary left mutations of \mathfrak{Q}_0 , we have

$$\begin{aligned}\gamma_1^{(6k+1)} &= -(3k+1)\gamma_1 - 3k(\gamma_2 + \gamma_3) = (-6k, -1, -\frac{1}{2}) \\ \gamma_2^{(6k+1)} &= \gamma_1 + \gamma_2 = (1, 0, 1) \\ \gamma_3^{(6k+1)} &= (3k+1)(\gamma_1 + \gamma_3) + 3k\gamma_2 = (6k+1, 1, -\frac{1}{2})\end{aligned}\tag{3.5}$$

and under $6k+1$ successive right ones, we get moreover

$$\begin{aligned}\gamma_1'^{(6k+1)} &= -(3k+1)\gamma_1 - 3k(\gamma_2 + \gamma_3) = (-6k, -1, -\frac{1}{2}) \\ \gamma_2'^{(6k+1)} &= (3k+1)(\gamma_1 + \gamma_2) + 3k\gamma_3 = (6k+1, 0, 1) \\ \gamma_3'^{(6k+1)} &= \gamma_1 + \gamma_3 = (1, 1, -\frac{1}{2})\end{aligned}\tag{3.6}$$

(iii) *the class $n=6k+2$*

The $(6k+2)$ left mutations of \mathfrak{Q}_0 give

$$\begin{aligned}\gamma_1^{(6k+2)} &= \gamma_3 = (1, 0, -1) \\ \gamma_2^{(6k+2)} &= (3k+2)\gamma_1 + (3k+1)(\gamma_2 + \gamma_3) = (6k+2, 1, \frac{1}{2}) \\ \gamma_3^{(6k+2)} &= -(3k+1)(\gamma_1 + \gamma_3) - 3k\gamma_2 = (-6k-1, -1, \frac{1}{2})\end{aligned}\tag{3.7}$$

and the $(6k+2)$ right ones to

$$\begin{aligned}\gamma_1^{(6k+2)} &= -(3k+1)(\gamma_1 + \gamma_3) - 3k\gamma_2 &= (-6k-1, -1, \frac{1}{2}) \\ \gamma_2^{(6k+2)} &= (3k+1)(\gamma_1 + \gamma_2) + (3k+2)\gamma_3 &= (6k+3, 0, -1) \\ \gamma_3^{(6k+2)} &= \gamma_1 &= (0, 1, \frac{1}{2})\end{aligned}\tag{3.8}$$

(iv) the class $n=6k+3$

Left mutations of \mathfrak{Q}_0 give

$$\begin{aligned}\gamma_1^{(6k+3)} &= (3k+2)(\gamma_1 + \gamma_3) + (3k+1)\gamma_2 &= (6k+3, 1, \frac{-1}{2}) \\ \gamma_2^{(6k+3)} &= -(3k+2)\gamma_1 - (3k+1)(\gamma_2 + \gamma_3) &= (-6k-2, -1, \frac{-1}{2}) \\ \gamma_3^{(6k+3)} &= \gamma_1 + \gamma_2 &= (1, 0, 1)\end{aligned}\tag{3.9}$$

and the right ones lead to

$$\begin{aligned}\gamma_1^{(6k+3)} &= -(3k+1)(\gamma_1 + \gamma_3) - (3k+2)\gamma_2 &= (-6k-3, 1, -\frac{1}{2}) \\ \gamma_2^{(6k+3)} &= (3k+1)\gamma_1 + (3k+2)(\gamma_2 + \gamma_3) &= (6k+4, -1, -\frac{1}{2}) \\ \gamma_3^{(6k+3)} &= \gamma_1 + \gamma_2 &= (1, 0, 1)\end{aligned}\tag{3.10}$$

(v) the class $n=6k+4$

In this sub-chamber, we have

$$\begin{aligned}\gamma_1^{(6k+4)} &= -(3k+2)(\gamma_1 + \gamma_3) - (3k+1)\gamma_2 &= (-6k-3, -1, \frac{1}{2}) \\ \gamma_2^{(6k+4)} &= \gamma_3 &= (1, 0, -1) \\ \gamma_3^{(6k+4)} &= (3k+3)\gamma_1 + (3k+2)(\gamma_2 + \gamma_3) &= (6k+4, 1, \frac{1}{2})\end{aligned}\tag{3.11}$$

and

$$\begin{aligned}\gamma_1^{(6k+4)} &= -(3k+2)(\gamma_1 + \gamma_2) - (3k+1)\gamma_3 &= (-6k-3, 0, -1) \\ \gamma_2^{(6k+4)} &= (3k+3)\gamma_1 + (3k+2)(\gamma_2 + \gamma_3) &= (6k+4, 1, \frac{1}{2}) \\ \gamma_3^{(6k+4)} &= \gamma_2 &= (1, -1, \frac{1}{2})\end{aligned}\tag{3.12}$$

(vi) the class $n=6k+5$

Under $6k+5$ left mutations of \mathfrak{Q}_0 , we have

$$\begin{aligned}\gamma_1^{(6k+5)} &= \gamma_1 + \gamma_2 &= (1, 0, 1) \\ \gamma_2^{(6k+5)} &= (3k+3)(\gamma_1 + \gamma_3) + (3k+2)\gamma_2 &= (6k+5, 1, \frac{-1}{2}) \\ \gamma_3^{(6k+5)} &= -(3k+3)\gamma_1 - (3k+2)(\gamma_2 + \gamma_3) &= (-6k-4, -1, \frac{-1}{2})\end{aligned}\tag{3.13}$$

and for the $6k+5$ right ones, we obtain

$$\begin{aligned}\gamma_1^{(6k+5)} &= -(3k+2)(\gamma_1 + \gamma_2) - (3k+3)\gamma_3 &= (-6k-5, 0, 1) \\ \gamma_2^{(6k+5)} &= (3k+3)(\gamma_1 + \gamma_3) + (3k+2)\gamma_2 &= (6k+5, 1, -\frac{1}{2}) \\ \gamma_3^{(6k+5)} &= \gamma_2 + \gamma_3 &= (2, -1, -\frac{1}{2})\end{aligned}\tag{3.14}$$

Below, we derive this BPS spectrum by using the quiver mutation group.

3.2 Deriving the BPS spectrum

Here we build the above BPS spectrum by using the group of quiver mutations. The EM charges of the 3 elementary BPS particles $\gamma_1, \gamma_2, \gamma_3$ are as in (3.1); they generate the 3-dimensional electric-magnetic lattice Γ_3 . These elementary BPS states play a crucial role in our analysis as they form the nodes of the primitive quiver \mathfrak{Q}_0 given by fig 1-(b). Recall that this quiver is one of the two basic object in the group theoretical approach; the other basic object is given by the set of mutation matrices \mathcal{M}_m . To get the BPS states of the weak coupling chamber of this supersymmetric gauge theory, we proceed in three steps as follows: **(1)** Introduce the vector $\Upsilon^{(0)}$ and the intersection matrix $\mathcal{J}^0 = \Upsilon^0 \circ \Upsilon^0$ describing \mathfrak{Q}_0 ; these are given by

$$\Upsilon^0 = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix}, \quad \mathcal{J}_{ij}^0 = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \quad (3.15)$$

(2) Perform successively the 3 following basic reflections

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad (3.16)$$

leading respectively to the 3 following mutated quivers $\mathfrak{Q}_1 = A\mathfrak{Q}_0$, $\mathfrak{Q}_2 = BA\mathfrak{Q}_0$, $\mathfrak{Q}_3 = CBA\mathfrak{Q}_0$. Like for the primitive \mathfrak{Q}_0 , these mutated quivers are described by the vectors $\Upsilon^{(n)}$ with $n = 1, 2, 3$; and the intersection matrices $\mathcal{J}_{ij}^{(n)}$; we denote these objects as

$$\Upsilon^{(n)} = \begin{pmatrix} \gamma_1^{(n)} \\ \gamma_2^{(n)} \\ \gamma_3^{(n)} \end{pmatrix}, \quad \mathcal{J}_{ij}^{(n)} = \gamma_i^{(n)} \circ \gamma_j^{(n)} \quad (3.17)$$

Now, using the following convention notation that will be justified later on

$$\mathcal{L}_1 = A, \quad \mathcal{L}_2 = BA, \quad \mathcal{L}_3 = CBA, \quad (3.18)$$

$$\mathcal{R}_1 = A, \quad \mathcal{R}_2 = AB, \quad \mathcal{R}_3 = ABC,$$

one can express (3.17) in the following form

$$\begin{aligned} \Upsilon^{(n)} &= \mathcal{L}_n \Upsilon^0, \quad \mathcal{J}^{(n)} = \mathcal{L}_n \mathcal{J}_{ij}^0 \mathcal{L}_n^T \\ \Upsilon'^{(n)} &= \mathcal{R}_n \Upsilon^0, \quad \mathcal{J}'^{(n)} = \mathcal{R}_n \mathcal{J}_{ij}^0 \mathcal{R}_n^T \end{aligned} \quad (3.19)$$

showing that all data about quiver mutations are captured by the \mathcal{L}_n 's and the \mathcal{R}_n 's. Notice that being reflections, we have

$$\begin{aligned} A^2 &= B^2 = C^2 = I_{id} \\ \det A &= \det B = \det C = -1 \end{aligned} \quad (3.20)$$

We also have

$$\begin{aligned} \mathcal{L}_2 &= \begin{pmatrix} 0 & 0 & 1 \\ 2 & 1 & 1 \\ -1 & 0 & -1 \end{pmatrix}, \quad \mathcal{R}_2 = \begin{pmatrix} -1 & 0 & -1 \\ 1 & 1 & 2 \\ 1 & 0 & 0 \end{pmatrix} \\ \mathcal{L}_3 &= \begin{pmatrix} 2 & 1 & 2 \\ -2 & -1 & -1 \\ 1 & 1 & 0 \end{pmatrix}, \quad \mathcal{R}_3 = \begin{pmatrix} -1 & -2 & -1 \\ 1 & 2 & 2 \\ 1 & 1 & 0 \end{pmatrix} \end{aligned} \quad (3.21)$$

satisfying some properties that follow from (3.20). In fact these properties are particular relations encoded by the group structure of the infinite set $\mathcal{G}_{weak} = \{\mathcal{L}_m, \mathcal{R}_m; m \in \mathbb{N}\}$ whose matrix representation will be given later. **(3)** Use the above particular quiver mutations to generate all others by distinguishing the two kinds of mutations:

(a) *Left mutations* generated by the \mathcal{L}_n 's and act as

$$\Upsilon^{(0)} \rightarrow \Upsilon^{(1)} \rightarrow \Upsilon^{(2)} \rightarrow \Upsilon^{(3)} \rightarrow \Upsilon^{(4)} \rightarrow \dots \quad (3.22)$$

with $\Upsilon^{(n)} = \mathcal{L}_n \Upsilon^{(0)}$ and $\mathcal{J}^{(n)} = \mathcal{L}_n \mathcal{J}_{ij}^{(0)} \mathcal{L}_n^T$ for any positive integer.

(b) *Right mutations* generated by the \mathcal{R}_n 's and operate like

$$\Upsilon^{(0)} \rightarrow \Upsilon'^{(1)} \rightarrow \Upsilon'^{(2)} \rightarrow \Upsilon'^{(3)} \rightarrow \Upsilon'^{(4)} \rightarrow \dots \quad (3.23)$$

with $\Upsilon'^{(n)} = \mathcal{R}_n \Upsilon^{(0)}$ and $\mathcal{J}'^{(n)} = \mathcal{R}_n \mathcal{J}_{ij}^{(0)} \mathcal{R}_n^T$ for any positive integer.

These sequences of quiver mutations combine together to form the following infinite set

left sector	right sector
$\mathcal{L}_{3n} = (CBA)^n$	$\mathcal{R}_{3n} = (ABC)^n$
$\mathcal{L}_{3n+1} = A\mathcal{L}_{3n}$	$\mathcal{R}_{3n+1} = \mathcal{R}_{3n}A$
$\mathcal{L}_{3n+2} = BA\mathcal{L}_{3n}$	$\mathcal{R}_{3n+2} = \mathcal{R}_{3n}AB$

(3.24)

that turns out to be an infinite discrete subgroup \mathcal{G}_{weak} of $GL(3, \mathbb{Z})$. The matrix realization of the group \mathcal{G}_{weak} gives exactly the electric-magnetic charges of the BPS states of the weak coupling chamber of this supersymmetric gauge theory. Let us compute explicitly this spectrum.

3.3 Explicit computation of the BPS spectrum

The study of the matrix representation of the quiver mutation group \mathcal{G}_{weak} reveals that the explicit realization of the \mathcal{L}_m and \mathcal{R}_m matrices can split into $\mathbb{Z}/6\mathbb{Z}$ classes as follows:

$$\begin{array}{ll}
\text{left sector} & \text{right sector} \\
\mathcal{L}_{6k} & = (\mathcal{L}_3)^{2k} & \mathcal{R}_{6k} & = (\mathcal{R}_3)^{2k} \\
\mathcal{L}_{6k+1} & = A\mathcal{L}_{6k} & \mathcal{R}_{6k+1} & = \mathcal{R}_{6k}A \\
\mathcal{L}_{6k+2} & = B\mathcal{L}_{6k+1} & \mathcal{R}_{6k+2} & = \mathcal{R}_{6k+1}B \\
\mathcal{L}_{6k+3} & = (\mathcal{L}_3)^{2k+1} & \mathcal{R}_{6k+3} & = (\mathcal{R}_3)^{2k+1} \\
\mathcal{L}_{6k+4} & = A\mathcal{L}_{6k+3} & \mathcal{R}_{6k+4} & = \mathcal{R}_{6k+3}A \\
\mathcal{L}_{6k+5} & = B\mathcal{L}_{6k+4} & \mathcal{R}_{6k+5} & = \mathcal{R}_{6k+4}B
\end{array} \tag{3.25}$$

with k a positive integer. Using the properties on the generators, one can show that the set $\{\mathcal{L}_n, \mathcal{R}_m; n, m \in \mathbb{N}\}$ has a discrete group structure. Moreover, using (3.16,3.25), we obtain:

(i) Left mutations

These mutations involves 6 infinite discrete series related to each other by the basic reflections. The first series is given by the following set of 3×3 matrices,

$$\mathcal{L}_{6k} = \begin{pmatrix} 1+3k & 3k & 3k \\ -3k & 1-3k & -3k \\ 0 & 0 & 1 \end{pmatrix}, \quad k \in \mathbb{N} \tag{3.26}$$

containing the identity $\mathcal{L}_0 = I_{id}$. Notice that the matrix elements of these series have the property $\det \mathcal{L}_{6k} = 1 \forall k$; but do not form a subgroup since the inverse of these matrices $(\mathcal{L}_{6k})^{-1}$ do not belong to the set $\{\mathcal{L}_{6k}, k \in \mathbb{N}\}$. As we will show later on, the inverse belong to the right mutations set; a property that explains the need of both left and right mutations noticed in [1, 2] to compute the full BPS spectrum. The other elements of the set of left mutations are given by eqs(3.25). Setting $\mathcal{B} = \mathcal{L}_{6k+1}$, $\mathcal{C} = \mathcal{L}_{6k+2}$, $\mathcal{D} = \mathcal{L}_{6k+3}$, $\mathcal{E} = \mathcal{L}_{6k+4}$, $\mathcal{F} = \mathcal{L}_{6k+5}$, we have:

$$\mathcal{B} = \begin{pmatrix} -3k-1 & -3k & -3k \\ 1 & 1 & 0 \\ 3k+1 & 3k & 3k+1 \end{pmatrix} \tag{3.27}$$

and

$$\mathcal{C} = \begin{pmatrix} 0 & 0 & 1 \\ 3k+2 & 3k+1 & 3k+1 \\ -3k-1 & -3k & -3k-1 \end{pmatrix}$$

$$\mathcal{D} = \begin{pmatrix} 3k+2 & 3k+1 & 3k+2 \\ -3k-2 & -3k-1 & -3k-1 \\ 1 & 1 & 0 \end{pmatrix}$$

as well as

$$\mathcal{E} = \begin{pmatrix} -3k-2 & -3k-1 & -3k-2 \\ 0 & 0 & 1 \\ 3k+3 & 3k+2 & 3k+2 \end{pmatrix}$$

$$\mathcal{F} = \begin{pmatrix} 1 & 1 & 0 \\ 3k+3 & 3k+2 & 3k+3 \\ -3k-3 & -3k-2 & -3k-2 \end{pmatrix}$$

with the properties $\det \mathcal{L}_{6k+1} = \det \mathcal{L}_{6k+3} = \mathcal{L}_{6k+5} = -1$; and $\det \mathcal{L}_{6k+2} = \mathcal{L}_{6k+4} = 1$.

(ii) *Right mutations*

Right mutations, contributing to the building of the weak chamber of BPS states of the supersymmetric $SU(2)$ gauge theory with a quark hypermultiplet, involve 6 infinite series related to each other by the basic A, B, C reflections. The first series is given by the set $\{\mathcal{R}_{6k}, k \in \mathbb{N}\}$ with $\det \mathcal{R}_{6k} = 1$; and \mathcal{R}_{6k} exactly the inverse of \mathcal{L}_{6k} for any positive integer k . This property shows that $\{\mathcal{L}_{6k}, \mathcal{R}_{6k}, k \in \mathbb{N}\}$ form a subgroup of the full set of quiver mutations \mathcal{G}_{weak} . The full set of right mutations is given by

$$\begin{aligned} \mathcal{A}' &= \mathcal{R}_{6k}, & \mathcal{C}' &= \mathcal{R}_{6k+2}, & \mathcal{E}' &= \mathcal{R}_{6k+4} \\ \mathcal{B}' &= \mathcal{R}_{6k+1}, & \mathcal{D}' &= \mathcal{R}_{6k+3}, & \mathcal{F}' &= \mathcal{R}_{6k+5} \end{aligned} \tag{3.28}$$

The explicit expression of these matrices are as follows

$$\mathcal{A}' = \begin{pmatrix} 1-3k & -3k & -3k \\ 3k & 1+3k & 3k \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathcal{B}' = \begin{pmatrix} -3k-1 & -3k & -3k \\ 3k+1 & 3k+1 & 3k \\ 1 & 0 & 1 \end{pmatrix}$$

$$\mathcal{C}' = \begin{pmatrix} -3k-1 & -3k & -3k-1 \\ 3k+1 & 3k+1 & 3k+2 \\ 1 & 0 & 0 \end{pmatrix}$$

and

$$\mathcal{F}' = \begin{pmatrix} -3k-2 & -3k-2 & -3k-3 \\ 3k+3 & 3k+2 & 3k+3 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\mathcal{E}' = \begin{pmatrix} -3k-2 & -3k-2 & -3k-1 \\ 3k+3 & 3k+2 & 3k+2 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\mathcal{D}' = \begin{pmatrix} -3k-1 & -3k-2 & -3k-1 \\ 3k+1 & 3k+2 & 3k+2 \\ 1 & 1 & 0 \end{pmatrix}$$

The rows of the matrices \mathcal{L}_{6m} and \mathcal{R}_m give exactly the electric-magnetic charges $\gamma_1^{(m)}$, $\gamma_2^{(m)}$, $\gamma_3^{(m)}$ of the BPS states of the weak coupling chamber. In the end of this section, notice that like in the pure SU(2) gauge theory, here also the number of the BPS states of the weak coupling chamber grows linearly with respect to the integer k ; so by defining the infinite the limit of the mutation matrices like

$$\mathcal{M}_\infty = \lim_{m \rightarrow \infty} \frac{1}{m} \mathcal{M}_m \quad (3.29)$$

we end with singular matrices that are associated with the BPS gauge particles. For instance, we have for the mutation matrix (3.26) the following singular limit:

$$\lim_{k \rightarrow \infty} \frac{1}{3k} \mathcal{L}_{6k} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \det \mathcal{L}_\infty = 0 \quad (3.30)$$

describing, in addition to the hypermultiplet with charge γ_3 , two BPS vector particles with electric-magnetic charges $\pm w$ with $w = \gamma_1 + \gamma_2 + \gamma_3$ which, by help of (3.1), is equal to $(2, 0, 0)$.

4 $SU(2)$ model with two hypermultiplets

The primitive quiver \mathfrak{Q}_0 of the $\mathcal{N} = 2$ supersymmetric $SU(2)$ gauge model with two quark hypermultiplets involves 4 particles with electric-magnetic $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ as depicted on fig 1-(c). The electric-magnetic charges are as follows

$$\gamma_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ -\frac{1}{2} \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ \frac{1}{2} \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 0 \\ 1 \\ \frac{1}{2} \\ 0 \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} 0 \\ 1 \\ -\frac{1}{2} \\ 0 \end{pmatrix} \quad (4.1)$$

and the intersection matrix \mathcal{J}_{ij}^0 , computed by help of the electric-magnetic products of the γ_i 's, is as given below. The extra third and fourth entries of these EM charge vectors refer to the $U_f(1) \times U'_f(1)$ charges of the flavor symmetry of the gauge model namely $SO_f(4) \sim SU_f(2) \times SU'_f(2)$. In our group theoretical set up, the primitive quiver is described by the pair $(\Upsilon^0, \mathcal{J}_{ij}^0)$ with

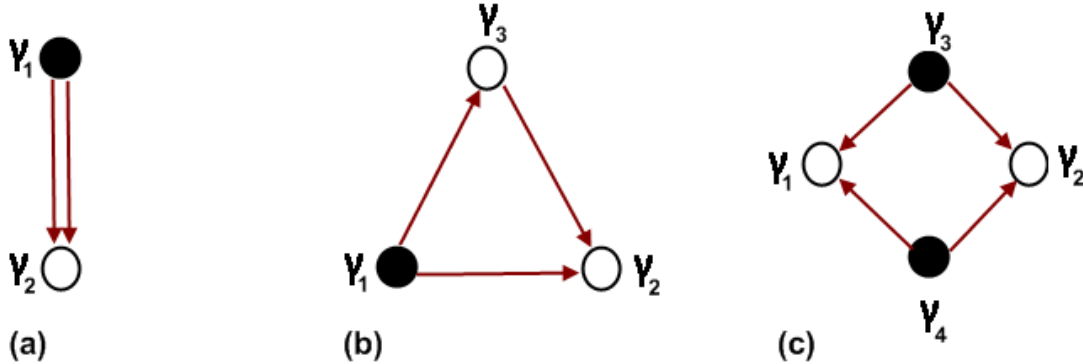


Figure 1: Primitive quivers: (a) $\mathcal{N} = 2$ pure $SU(2)$ gauge model, (b) adding one quark hypermultiplet, (c) adding two hypermultiplets.

$$\Upsilon^0 = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{pmatrix}, \quad \mathcal{J}_{ij}^0 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ -1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{pmatrix} \quad (4.2)$$

The BPS spectrum of the weak coupling chamber of this supersymmetric gauge theory with two quark hypermultiplets can be obtained by extending the group theoretical

method used in the previous sections. In this case also, the quiver mutation group of the weak coupling chamber is an infinite discrete set $\mathcal{G}_{weak} = \{\mathcal{L}_m, \mathcal{R}_m; m \in \mathbb{N}\}$ generated by the two following reflections

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \quad (4.3)$$

satisfying $A^2 = B^2 = I_{id}$. As in previous examples, here also the weak coupling chamber has several sub-chambers given by the four sequences $\{\mathcal{L}_{4k+q}; \mathcal{R}_{4k+q}\}$ with $q = 0, 1, 2, 3$. The general terms of these sequences are as follows,

$$\begin{aligned} \mathcal{L}_{4k} &= (BA)^{2k}, & \mathcal{L}_{4k+1} &= A\mathcal{L}_{4k}, & \mathcal{L}_{4k+2} &= B\mathcal{L}_{4k+1}, & \mathcal{L}_{4k+3} &= A\mathcal{L}_{4k+2} \\ \mathcal{R}_{4k} &= (AB)^{2k}, & \mathcal{R}_{4k+1} &= B\mathcal{R}_{4k}, & \mathcal{R}_{4k+2} &= A\mathcal{R}_{4k+1}, & \mathcal{R}_{4k+3} &= B\mathcal{R}_{4k+2} \end{aligned} \quad (4.4)$$

Clearly this infinite set \mathcal{G}_{weak} has a discrete group structure with identity element given by the 4×4 identity matrix; in fact \mathcal{G}_{weak} is a particular subgroup of $GL(4, \mathbb{Z})$. Using the matrix expressions of the reflections A and B, we obtain for the \mathcal{L}_{4k} left mutations

$$\mathcal{L}_{4k} = \begin{pmatrix} 1-2k & -2k & -2k & -2k \\ -2k & 1-2k & -2k & -2k \\ 2k & 2k & 2k+1 & 2k \\ 2k & 2k & 2k & 2k+1 \end{pmatrix} \quad (4.5)$$

and for the corresponding right mutations

$$\mathcal{R}_{4k} = \begin{pmatrix} 2k+1 & 2k & 2k & 2k \\ 2k & 2k+1 & 2k & 2k \\ -2k & -2k & 1-2k & -2k \\ -2k & -2k & -2k & 1-2k \end{pmatrix} \quad (4.6)$$

which is exactly $(\mathcal{L}_{4k})^{-1}$. The expressions of the other mutations matrices are directly learnt from (4.4-4.3). After doing these computations, we can determine the electric-magnetic charges of the BPS and anti-BPS states from the rows of the various mutation matrices. For BPS states, we find the following spectrum

$$\begin{aligned} 2k\gamma_1 + (2k+1)(\gamma_2 + \gamma_3 + \gamma_4) & , & (2k+1)\gamma_1 + 2k(\gamma_2 + \gamma_3 + \gamma_4) \\ 2k\gamma_2 + (2k+1)(\gamma_1 + \gamma_3 + \gamma_4) & , & (2k+1)\gamma_2 + 2k(\gamma_1 + \gamma_3 + \gamma_4) \\ (2k+1)\gamma_1 + (2k+2)(\gamma_2 + \gamma_3 + \gamma_4) & , & (2k+1)\gamma_3 + 2k(\gamma_1 + \gamma_2 + \gamma_4) \\ (2k+1)\gamma_2 + (2k+2)(\gamma_1 + \gamma_3 + \gamma_4) & , & (2k+1)\gamma_4 + 2k(\gamma_1 + \gamma_2 + \gamma_3) \end{aligned} \quad (4.7)$$

and

$$\begin{aligned} (2k+2)\gamma_1 + (2k+1)(\gamma_2 + \gamma_3 + \gamma_4) \\ (2k+2)\gamma_2 + (2k+1)(\gamma_1 + \gamma_3 + \gamma_4) \\ (2k+2)\gamma_3 + (2k+1)(\gamma_1 + \gamma_2 + \gamma_4) \end{aligned} \tag{4.8}$$

$$(2k+2)\gamma_4 + (2k+1)(\gamma_1 + \gamma_2 + \gamma_3) \tag{4.9}$$

The corresponding anti-BPS states have opposite electric-magnetic charges. In the end of this study, notice that the BPS spectrum grows linearly with integer k . For the limit $k \rightarrow \infty$, one ends with singular matrices as shown on the following example

$$\lim_{k \rightarrow \infty} \frac{1}{2k} \mathcal{L}_{4k} = \begin{pmatrix} -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad \det \mathcal{L}_\infty = 0 \tag{4.10}$$

This singular limit describes BPS vector states with electric magnetic charge vectors as $\pm w$ with $w = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$; which by help of (4.1) is noting but the charge $(2, 0, 0, 0)$ of the massive vector bosons of the spontaneously broken $SU(2)$ gauge symmetry.

5 Conclusion and comment

In this paper, we have studied the weak coupling chamber of some models of $\mathcal{N} = 2$ supersymmetric $SU(2)$ gauge theory by using the symmetry structure of the set of quiver mutations of the BPS quiver approach of [1, 2]. Generally, the quiver mutations of $\mathcal{N} = 2$ supersymmetric theories are discrete symmetries of BPS chambers; and may form either a finite set or an infinite one. Finite quiver mutations concern the strong coupling chambers of supersymmetric QFT's as shown in [11]; these discrete mutations turn out to be isomorphic to a class of (dihedral) subgroups of the well known finite Coxeter groups. The latters are used in building of the Dynkin diagrams of finite dimensional Lie algebra of the gauge symmetries. Infinite mutations are symmetries of weak coupling chambers of supersymmetric QFTs. The present study has been done for the case of $SU(2)$ gauge theory without and with fundamental matter; but this construction can be also applied for $\mathcal{N} = 2$ supersymmetric QFTs with higher dimensional gauge symmetries.

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